Inequalities and Identities.
A Journey into Mathematics

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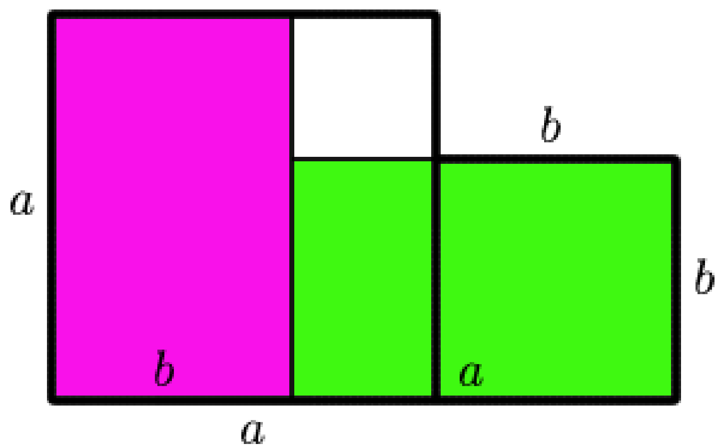

University of Craiova, Department of Mathematics

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Mostar, November 11-15, 2015.

1. Augustin-Louis Cauchy, 1821

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. *ANALYSE ALGÉBRIQUE.*

DE L'IMPRIMERIE ROYALE.

Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821

Page 456, formula (31):

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} a_i b_i\right)^2 + \sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2.$$

Consequence: Cauchy-Buniakovski-Schwarz Inequality and its equality case.

2. History

A particular form of Lagrange's identity: Fibonacci in 1225 in his *Liber Quadratorum* (*Book of Squares*):

$$\left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right) = (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2$$
$$= (a_1 b_1 - a_2 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 .$$

For integer values of the variables, this means that the product of sums of squares is again a sum of squares (a fact that originates to Book 13, Problem 19, in *Arithmetica* of Diophantus of Alexandria):

$$(1^2 + 4^2)(2^2 + 7^2) = 30^2 + 1^2 = 26^2 + 15^2.$$

Modern proof via complex number multiplication:

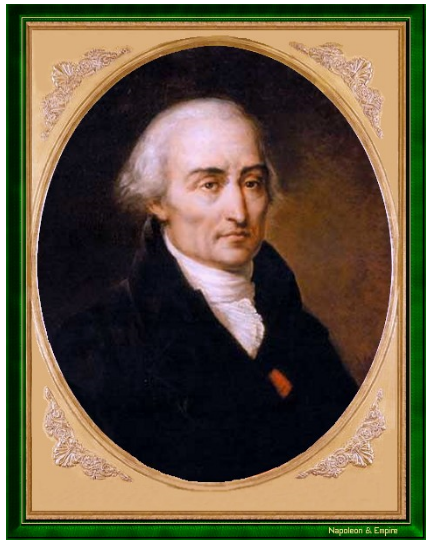
$$|a_1 + i a_2|^2 \, |b_1 + i b_2|^2 = |(a_1 + i a_2)(b_1 + i b_2)|^2 .$$

Brahmagupta (598–668): $\left(a_1^2 - N a_2^2\right)\left(b_1^2 - N b_2^2\right)$ equals

$$(a_1 b_1 + N a_2 b_2)^2 - N (a_1 b_2 + a_2 b_1)^2$$

Bhaskara II in 1150 solved Pell's equation $x^2 - N y^2 = 1$.

3. Lagrange's Algebraic Identity



1773, *Quelques problémes sur les pyramides triangulaires*, p. 663, lines 6-8:

$$\left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right) = \left(\sum_{i=1}^{3} a_i b_i\right)^2 + \sum_{1\le i<j\le 3} (a_i b_j - a_j b_i)^2.$$

In other words

$$\|u\|^2\,\|v\|^2 = |\langle u, v\rangle|^2 + \|u \times v\|^2 \quad \text{for all } u, v \in \mathbb{R}^3.$$

4. Lagrange's Barycentric Identity

1783, *Sur une nouvelle proprieté du centre de gravité*:

$$(L)\ \frac{1}{M}\sum_{k=1}^{n} m_k \, \|z - x_k\|^2$$
$$= \left\| z - \frac{1}{M}\sum_{k=1}^{n} m_k x_k \right\|^2 + \frac{1}{M^2}\sum_{1\le i<j\le n} m_i m_j \, \|x_i - x_j\|^2.$$

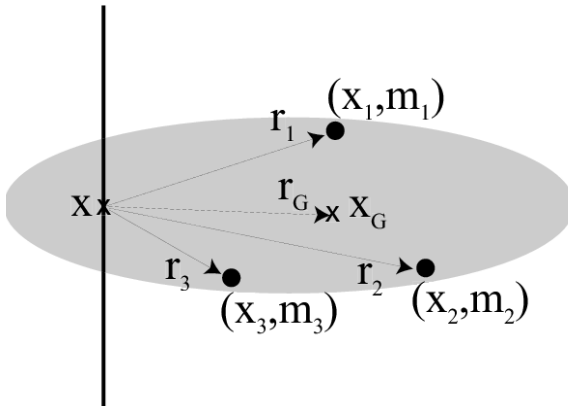
For $z = 0$, $m_k = p_k a_k^2$ and $x_k = y_k / a_k$ in (L) :

$$\left(\sum_{k=1}^{n} p_k a_k^2\right)\left(\sum_{k=1}^{n} p_k \|y_k\|^2\right)$$
$$= \left\| \sum_{k=1}^{n} p_k a_k y_k \right\|^2 + \sum_{1\le i<j\le n} p_i p_j \|a_j y_i - a_i y_j\|^2.$$

Consequence:

$$\left\| \sum_{k=1}^{n} p_k a_k y_k \right\|^2 \le \left(\sum_{k=1}^{n} p_k a_k^2\right)\left(\sum_{k=1}^{n} p_k \|y_k\|^2\right).$$

5. Mechanical Interpretation



1673 Christiaan Huygens: the parallel axis theorem,

$$I_x = I_{com} + Md^2.$$

Equivalently (Huygens-Steiner identity),

$$\frac{1}{M}\sum_{k=1}^{n} m_k \|x - x_k\|^2$$
$$= ||z - \frac{1}{M}\sum_{k=1}^{n} m_k x_k||^2 + \frac{1}{M}\sum_{k=1}^{n} m_k \|x_k - \frac{1}{M}\sum_{j=1}^{n} m_j x_j\|^2.$$

1765 Leonhard Euler, *Theoria motus corporum solidorum seu rigidorum.*

A connection to metric geometry

Consider a triangle ΔABC with side lengths a, b, c, the centroid G , the center of the circumscribed circle O, the center of the inscribed circle I, etc.

Lagrange's barycentric identity (applied to the family of vertices of the triangle, with equal weights $m_1 = m_2 = m_3 = 1$, and for the choice of x as the center of the circumscribed circle) yields the following formula for the radius of circumscribed circle (attributed to Leibniz):

$$R^2 = OG^2 + \frac{1}{9}(a^2 + b^2 + c^2).$$

As a consequence, $a^2 + b^2 + c^2 \leq 9R^2$, with equality if (and only if) the triangle is equilateral.

More generally, if R is the radius of the smallest ball containing a finite family of points $x_1, \ldots, x_n \in \mathbb{R}^N$, then

$$\frac{1}{n}\left(\sum_{i<j} \|x_i - x_j\|^2\right)^{1/2} \leq R.$$

The formula

$$R^2 = OI^2 + 2Rr \quad \text{(equivalently } OI^2 = R(R-2r)\text{)}$$

was discovered independently by W. Chapple (1746) and L. Euler (1765). It implies

$$2r \leq R,$$

with equality if (and only if) the triangle is equilateral.

This follows from the Huygens-Steiner identity applied to the family of vertices of the triangle, the weights

$m_1 = a/(a{+}b{+}c)$, $m_2 = b/(a{+}b{+}c)$ and $m_3 = c/(a{+}b{+}c)$,

and for x the center of the circumscribed circle.

Basic clue: the barycenter is

$$\frac{1}{M}\sum_{k=1}^{n} m_k x_k = I,$$

the center of the *inscribed* circle.

6. Weighted Least Squares

Given a family of points $x_1, ..., x_n$ *in* $\mathbb{R}^N$ *and real weights* $m_1, ..., m_n \in \mathbb{R}$ with $M = \sum_{k=1}^{n} m_k > 0$, *then*

$$\min_{x \in \mathbb{R}^N} \sum_{k=1}^{n} m_k \|x - x_k\|^2 = \frac{1}{M} \cdot \sum_{i<j} m_i m_j \|x_i - x_j\|^2.$$

The minimum is attained at one point,

$$x_G = \frac{1}{M} \sum_{k=1}^{n} m_k x_k.$$

Giulio Carlo Fagnano: the existence of a point P in the plane of a triangle ABC that minimizes the sum $PA^2 + PB^2 + PC^2$.

Carl Friedrich Gauss: the foundations of the least-squares analysis in 1795.

1809 Theory of motion of the celestial bodies moving in conic sections around the Sun.

7. Poincaré's Inequality (1890)

If Ω is a bounded connected set in $\mathbb{R}^N$ with Lipschitz boundary, then there exists a constant C, depending only on Ω and N, such that for every $f \in H^1(\Omega)$,

$$\left\| f - \frac{1}{vol\Omega} \int_\Omega f dx \right\|_{L^2} \leq C \left\| \nabla f \right\|_{L^2}.$$

Starting point: If μ is a probability measure on a space Ω and f and g are two real random variables belonging to the space $L^2(\mu)$:

$$\left(\int_\Omega f^2 d\mu \right) \left(\int_\Omega g^2 d\mu \right) - \left(\int_\Omega fg d\mu \right)^2$$
$$= \frac{1}{2} \int_\Omega \int_\Omega \left(f(x)g(y) - f(y)g(x) \right)^2 d\mu(x) d\mu(y),$$

Special case of smooth functions $f : [0,1] \to \mathbb{R}$ that verify the condition $\int_0^1 f dx = 0$:

$$\int_0^1 f^2 dx = \frac{1}{2} \int_0^1 \int_0^1 (f(x) - f(y))^2 \, dx dy.$$

By taking Lagrange's mean value theorem (with integral remainder) we infer that

$$f(x) = f(y) + (x - y) \int_0^1 f'(tx + (1 - t)y) dt,$$

whence one can conclude that

$$\int_0^1 f^2 dx \leq \frac{1}{2} \int_0^1 f'^2(s) ds.$$

8. The isoperimetric problem

Steiner

Weierstrass

Hurwitz

Wirtinger

Dido's Problem: What is the closed curve which has the maximum area for a given perimeter ?

If $\mathcal{C}$ is a simple closed smooth curve given parametrically, with $L = \int_{\mathcal{C}} ds$ its arc length and $A = -\int_{\mathcal{C}} y dx$ the area enclosed by $\mathcal{C}$, then

$$L^2 \geq 4\pi A;$$

equality holds if and only if $\mathcal{C}$ is a circle.

Solution: Parametrize $\mathcal{C}$ with constant speed $L/2\pi$. By a translation we may also assume $\int_0^{2\pi} y dt = 0$. Using Green's formula,

$$L^2 - 4\pi A = 2\pi \int_0^{2\pi} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2y\frac{dx}{dt} \right] dt$$

$$= 2\pi \int_0^{2\pi} \left[\frac{dx}{dt} + y\right]^2 dt + 2\pi \int_0^{2\pi} \left[\left(\frac{dy}{dt}\right)^2 - y^2 \right] dt.$$

Wilhelm Wirtinger (1904): Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π, which is C^1 and such that $\int_0^{2\pi} f dx = 0$. Then

$$\int_0^{2\pi} f^2 dx \leq \int_0^{2\pi} f'^2(s) ds$$

with equality if and only if $f(x) = a \sin(x) + b \cos(x)$.

Proof. By Parseval's identity,

$$\int_0^{2\pi} f^2 dx = \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

and

$$\int_0^{2\pi} f'^2 dx = \sum_{n=1}^{\infty} n \left(a_n^2 + b_n^2\right).$$

9. The uncertainty principle

Hermann Weyl, *Theory of groups and Quantum Mechanics*, Dover, 1950: One can not jointly localize a signal in time and frequency arbitrarily well; either one has poor frequency localization or poor time localization.

Suppose that $f(t)$ is a finite energy signal with Fourier transform $F(\omega)$ and $\sqrt{|t|}f(t) \to 0$ as $|t| \to \infty$. Then

$$Dd \geq \frac{1}{2}$$

where $E = \int_{\mathbb{R}} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} |F(\omega)|^2 \, d\omega$,

$$d^2 = \frac{1}{E} \int_{\mathbb{R}} t^2 \, |f(t)|^2 \, dt \text{ and } D^2 = \frac{1}{2\pi E} \int_{\mathbb{R}} \omega^2 \, |F(\omega)|^2 \, dt.$$

Moreover, equality holds only if $f(t)$ has the form

$$f(t) = Ce^{-\alpha t^2}.$$

Proof. Suppose that f is real. Notice that

$$\left|\int_{-\infty}^{\infty} tf(t)f'(t)dt\right|^2 \leq \int_{-\infty}^{\infty} t^2 f^2(t)dt \int_{-\infty}^{\infty} f'^2(t)dt$$

and

$$\int_{-\infty}^{\infty} tf(t)f'(t)dt = t\frac{f^2(t)}{2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}f^2(t)dt = -\frac{1}{2}E.$$

By Parseval's Theorem,

$$\int_{-\infty}^{\infty} f'^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 \left|F(\omega)\right|^2 d\omega.$$

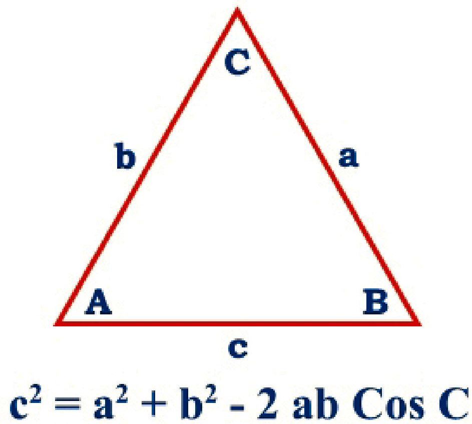
Therefore

$$\frac{1}{4}E^2 \leq d^2E \times D^2E,$$

that is,

$$\frac{1}{2} \leq dD.$$

10. CBS Inequality is just a consequence of Law of Cosine



For every two nonzero vectors x and y in a real inner product space H,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = 2 - 2\frac{\langle x, y\rangle}{\|x\| \, \|y\|}.$$

11. Convexity

J.L.W.V. Jensen

O. Hölder

O. Hölder (1889): Assume $m \leq f'' \leq M$. Then there is $\mu \in [m, M]$ such that

$$\sum_{k=1}^{n} \lambda_k f(x_k) - f\left(\sum_{k=1}^{n} \lambda_k x_k\right) = \frac{1}{4}\mu \sum_{jJ=1}^{n} \sum_{k=1}^{n} \lambda_j \lambda_k (x_j - x_k)^2.$$

for all $x_1, x_2, ..., , x_n \in [a, b]$ and all $\lambda_1, \lambda_2, ..., \lambda_n \in [0, 1]$ with $\sum \lambda_k = 1$.

J. L. W. V. Jensen (1906): the general case of convex functions.

12. The classical Hermite-Hadamard inequality

Ch. Hermite

J. Hadamard

If $f : [a,b] \longrightarrow \mathbb{R}$ is convex,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Each inequality characterizes convexity.

Equality only for affine functions.

The Hermite-Hadamard double inequality follows from two identities related to trapezoidal and midpoint rules of quadratures:

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b \frac{(b-x)(x-a)}{2} f''(x)dx$$

and

$$\frac{1}{b-a}\int_a^b f(x)dx = f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b \varphi(x) f''(x)dx,$$

where

$$\varphi(x) = \begin{cases} \frac{(x-a)^2}{2} & \text{if } x \in [a, (a+b)/2] \\ \frac{(b-x)^2}{2} & \text{if } x \in [(a+b)/2, b]. \end{cases}$$

Higher order rules [**1**] yield Hermite-Hadamard type inequalities for convex functions of n-th order.

Positive Polynomials and Sums of Squares

D. Hilbert

Every nonnegative polynomial of a *single* variable can be expressed as a sum of squares (sos) of polynomials.

Basic idea:

$$c^2 \prod_{j=1}^{r} (t - t_j)^{2m_j} \prod_{k=1}^{s} (t - (\alpha_k + i\beta_k))\,(t - (\alpha_k - i\beta_k))$$
$$= Q^2(t) \prod_{k=1}^{s} ((t - \alpha_k)^2 + \beta_k^2) = R^2(t) + S^2(t),$$

via Fibonacci's identity.

The several variables case

Special cases when nonnegative polynomials are sums of squares

Hilbert (1888): quadratic polynomials in any number of variables; quartic polynomial in 2 variables.

A. Hurwitz (1891):

$$\frac{x_1^{2n}+x_2^{2n}+\cdots+x_n^{2n}}{n}-x_1^2x_2^2\cdots x_n^2 = \text{sum of squares.}$$

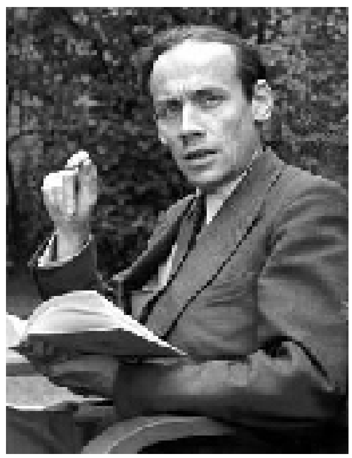
For example:

$$\frac{x_1^4+x_2^4+x_3^4+x_4^4}{4}-x_1x_2x_3x_4 = \frac{\left(x_1^2-x_2^2\right)^2+(x_3^2-x_4^2)^2}{4}+\frac{(x_1x_2-x_3x_4)^2}{2}.$$

P. E. Frenkel and P. Horvath, *Minkowski's inequality and sums of squares*, ArXiv 1206.5783v2 /4 January 2013

13. Hilbert's Seventeenth Problem

Given a multivariate polynomial that takes only nonnegative values over the reals, can it be represented as a sum of squares of rational functions? Yes, Artin (1927)



Motzkin (1966) :

$$
\begin{aligned}
&1 + x^4y^2 + x^2y^4 - 3x^2y^2 \\
&\quad = \left(\frac{x^2y(x^2+y^2-2)}{x^2+y^2}\right)^2 + \left(\frac{xy^2(x^2+y^2-2)}{x^2+y^2}\right)^2 \\
&\qquad\qquad + \left(\frac{xy(x^2+y^2-2)}{x^2+y^2}\right)^2 + \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 .
\end{aligned}
$$

14. Two Open Problems

Is every inequality the consequence of an identity?

Find new identities and their associated inequalities.

15. A Generalization of Lagrange's Identity

Lemma 1. (N&Stephan [**14**], [**15**]) *Let I be an interval of $\mathbb{R}$ endowed with a discrete measure $\mu = \sum_{k=1}^{n} p_k \delta_{x_k}$, whose weights p_k are all nonzero and sum to 1. We assume that the* barycenter *of μ,*

$$b_\mu = \sum_{k=1}^{n} p_k x_k,$$

belongs to $I \setminus \{x_1, x_2, ..., x_n\}$. Then every function $f : I \to \mathbb{R}$ verifies the identity

$$\sum_{k=1}^{n} p_k f(x_k)$$
$$= f\left(b_\mu\right) + \sum_{1 \le k < j \le n} p_k p_j \left(s(x_k) - s(x_j)\right)\left(x_k - x_j\right),$$

where

$$s(x) = \frac{f(x) - f(b_\mu)}{x - b_\mu} \quad \textit{for } x \in I \setminus \{b_\mu\}$$

is the slope function of the segment joining the points of abscissas x and b_μ.

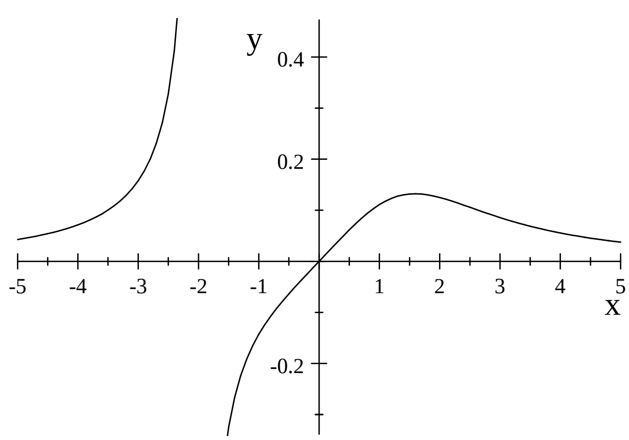
16. A Math. Olympiad Problem

Let $a, b, c, d \geq 0$ and $a + b + c + d = 4$. Show

$$\sum \frac{a}{a^3 + 8} \leq \frac{4}{9}.$$

Partial solution. The function $F(x) = \frac{x}{x^3+8}$ is concave for $x \in [0, 2]$. According to Jensen's inequality, for $a, b, c, d \in [0, 2]$ and $a + b + c + d = 4$, we have

$$\frac{1}{4}\left(\sum \frac{a}{a^3 + 8}\right) \leq \frac{\frac{a+b+c+d}{4}}{\left(\frac{a+b+c+d}{4}\right)^3 + 8} = \frac{1}{9}.$$



Theorem 1. *(Jensen's Inequality for Mixed Convex Functions). Assume that all weights p_k are nonnegative. If $s(x)$ is increasing, then*

$$\sum_{k=1}^{n} p_k f(x_k) \geq f\left(b_\mu\right),$$

while if $s(x)$ is decreasing, then this inequality works in the reversed direction.

Solution to the problem. Apply Theorem 1 to the function F and the probability measure $\mu = \frac{1}{n}\sum\limits_{k=1}^{n} \delta_{x_k}$, where $x_1, x_2, ..., , x_n > -2$ and $\frac{1}{n}\sum\limits_{k=1}^{n} x_k = 1$. In this case $b_\mu = 1$ and the slope function

$$s(x) = \frac{F(x) - F(1)}{x - 1} = \frac{8 - x - x^2}{9(8 + x^3)}.$$

is decreasing on $(-2, \infty)$. Therefore

$$\max \frac{1}{n}\sum_{k=1}^{n} F\left(x_k\right) = \frac{1}{9}.$$

Other examples:

(a) Suppose that $x_1, ..., x_n > 0$ and $\frac{1}{n} \sum\limits_{k=1}^{n} x_k = e$. Then $\prod\limits_{k=1}^{n} x_k^{1/x_k} \leq e^{1/e}$.

(b) Kostant-Michor inequality (see [**3**]): Suppose that $x_1, ..., x_n$ are real numbers such that $\frac{1}{n} \sum_{k=1}^{n} x_k \geq 0$. Then

$$\sum_{k=1}^{n} x_k e^{x_k} \geq \frac{2}{n} \sum_{k=1}^{n} x_k^2.$$

17. The Several Variables Case

First Step: Adapt to several variables what we did in $\mathbb{R}$.

C a subset of the Euclidean space $\mathbb{R}^N$ endowed with a real measure $\mu = \sum_{i=1}^{n} p_i \delta_{x_i}$ whose weights p_i are all nonzero and sum up to 1. The *barycenter* of μ,

$$b_\mu = \sum_{i=1}^{n} p_i x_i,$$

is supposed to be in $C \setminus \{x_1, ..., x_n\}$.

Theorem 2. (N&Stephan [**14**], [**15**]) *Under the above assumptions on C and μ, every function $f : C \to \mathbb{R}$ verifies the following extension of Lagrange's identity*:

(GL)

$$\sum_{i=1}^{n} p_i f(x_i) = f(b_\mu) + \sum_{i<j} p_i p_j \left\langle s(x_i) - s(x_j), x_i - x_j \right\rangle,$$

where

$$s(x) = \frac{f(x) - f(b_\mu)}{\|x - b_\mu\|} \cdot \frac{x - b_\mu}{\|x - b_\mu\|} \quad \textit{for } x \in C \setminus \{b_\mu\}.$$

When f is a continuously differentiable function defined on a convex subset C of $\mathbb{R}^N$, one can state the identity (GL) in terms of gradients:

$$(SGL) \quad \sum_{i=1}^{n} p_i f(x_i) = f\left(b_\mu\right)$$
$$+ \sum_{i<j} p_i p_j \int_0^1 \langle \nabla f(P_i(t)) - \nabla f(P_j(t)), x_i - x_j \rangle dt,$$

where $P_i(t) = tx_i + (1-t)b_\mu$.

Note: (L) is the particular case where $f(x) = \frac{1}{2} \|x\|^2$, $x \in \mathbb{R}^N$. Indeed,

$$\nabla f(x) = x \quad \text{and} \quad \nabla^2 f(x) = I_n.$$

Example 1. *In the Euclidean spaces,*

$$6\left(\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2\right) + 2\|x_1 + x_2 + x_3\|^2$$
$$= 3\left(\|x_1 + x_2\|^2 + \|x_2 + x_3\|^2 + \|x_3 + x_1\|^2\right)$$
$$+ \sum_{1 \le i < j \le 3} \left\|x_i - x_j\right\|^2.$$

A consequence is the inequality

$$\frac{\|x_1\|^2+\|x_2\|^2+\|x_3\|^2}{3}+\left\|\frac{x_1+x_2+x_3}{3}\right\|^2$$
$$\geq \frac{2}{3}\left(\left\|\frac{x_1+x_2}{2}\right\|^2+\left\|\frac{x_2+x_3}{2}\right\|^2+\left\|\frac{x_3+x_1}{2}\right\|^2\right),$$

which illustrates the phenomenon of $(2D)$-*convexity*. See Mihail Bencze, C. P. Niculescu and Florin Popovici [**2**].

Example 2. (Hlawka' Identity) *In Euclidean spaces,*

$$\|x\|^2+\|y\|^2+\|z\|^2+\|x+y+z\|^2$$
$$=\|x+y\|^2+\|y+z\|^2+\|z+x\|^2,$$

Example 3. *Discrepancy between the weighted harmonic mean* $H=\left(\sum_{i=1}^{n}\frac{p_i}{x_i}\right)^{-1}$ *and the weighted arithmetic mean* $A=\sum_{i=1}^{n}p_ix_i$:

$$1+\frac{\sigma_\mu^2}{M^2}\leq\frac{A}{H}\leq 1+\frac{\sigma_\mu^2}{m^2}.$$

where

$$\sigma_\mu^2=\sum_{1\leq i<j\leq n}p_ip_j(x_i-x_j)^2$$

represents the variance *of the given family.*

18. A Second Generalization

Embedding Jensen's Inequality into an identity:

Theorem 3. (N&Stephan [**15**]). *Suppose that K is a Borel measurable convex subset of $\mathbb{R}^N$ (or more generally of a real Hilbert space), endowed with a real Borel measure μ such that $\mu(K) = 1$ and $b_\mu \in K$. Then for every function $f : K \to \mathbb{R}$ of class C^2 we have the identity*

$$f(b_\mu) + \frac{1}{2}\int_K \int_K \langle \nabla f(x) - \nabla f(y), x - y\rangle d\mu(x) d\mu(y)$$
$$= \int_K f(x) d\mu(x)$$
$$+ \int_K \int_0^1 (1-t) \langle \nabla^2 f(x + t(b_\mu - x)) (b_\mu - x), b_\mu - x\rangle dt d\mu(x),$$

provided that all integrals are legitimate.

In the particular case where $f(x) = \frac{1}{2}\|x\|^2$, $x \in \mathbb{R}^N$, we recover the identity (L).

Theorem 4. (N&Stephan [**15**]). *Suppose that K is a compact convex subset of the Euclidean space $\mathbb{R}^N$, endowed with a Borel probability measure μ, and f is a convex function of class C^1, defined on a neighborhood of K. Then*

$$\frac{1}{2}\int_K \int_K \langle \nabla f(x) - \nabla f(y), x - y\rangle d\mu(x) d\mu(y)$$
$$\geq \int_U f(x) d\mu(x) - f(b_\mu) \geq 0.$$

This provides a converse for each instance of Jensen's inequality.

Example: Hardy's inequality (1925) asserts that *if f is nonnegative and p-integrable on $(0, \infty)$, with $p > 1$, then*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx.$$

Extension by Y. Bicheng, Z. Zhuohua and L. Debnath (see also Persson and Samko. See [**18**], Theorem 2.1):

Theorem 5. *Suppose that h is a nonnegative and locally integrable function on $(0,\ell)$ (where $0<\ell\le\infty$) and $p\in(-\infty,0)\cup[1,\infty)$.* Then

$$\int_0^\ell \left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p \frac{dx}{x} \le \int_0^\ell h^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}.$$

Theorem 4 yields the following converse to Theorem 5:

Theorem 6. *If $u:(0,\infty)\to\mathbb{R}$ is a convex function of class C^1 and $h:(0,\ell)\to\mathbb{R}$ is a nonnegative integrable function, then*

$$\int_0^\ell u\left(\frac{1}{x}\int_0^x h(t)dt\right)\frac{dx}{x}+$$
$$\frac{1}{2}\int_0^\ell (\frac{1}{x^2}\int_0^x\int_0^x \Big(u'(h(s))-u'(h(t))\Big)(h(s)-h(t))\,dsdt)\frac{dx}{x}$$
$$\ge \int_0^\ell u(h(x))\left(1-\frac{x}{\ell}\right)\frac{dx}{x}.$$

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Thank You!